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On capacity and torsional rigidity

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Abstract

We investigate extremal properties of shape functionals which are products of Newtonian capacity $\mathrm{cap}\,(\overline{\Omega})$, and powers of the torsional rigidity $T(\Omega)$, for an open set $\Omega \subset \mathbb{R}^d$ with compact closure $\overline{\Omega}$, and prescribed Lebesgue measure. It is shown that if Ω is convex, then $\mathrm{cap}\,(\overline{\Omega})T^q(\Omega)$ is (i) bounded from above if and only if $q \geqslant 1$, and (ii) bounded from below and away from 0 if and only if $q \leqslant \frac{d-2}{2(d-1)}$. Moreover a convex maximiser for the product exists if either $q > 1$, or $d = 3$ and $q = 1$. A convex minimiser exists for $q < \frac{d-2}{2(d-1)}$. If $q \leqslant 0$, then the product is minimised among all bounded sets by a ball of measure 1.

1. *Introduction and main results*

Several classical inequalities of mathematical physics are of the following form. Let F and H be strictly positive set functions defined on a suitable collection $\mathfrak{C}$ of open sets in $\mathbb{R}^d$, and which satisfy scaling relations

$$F(t\Omega) = t^{\beta_1} F(\Omega), \quad H(t\Omega) = t^{\beta_2} H(\Omega), \quad t > 0,$$

where $t\Omega$ is homothety of Ω, and β_1, β_2 are constants. Then the shape functional

$$G(\Omega) = H(\Omega)F(\Omega)^{-\beta_2/\beta_1},$$

is invariant under homotheties, and in some cases this quantity is minimal (respectively, maximal) for some open set $\Omega^* \in \mathfrak{C}$,

$$G(\Omega) \geqslant G(\Omega^*) \quad (\text{respectively}, G(\Omega) \leqslant G(\Omega^*)), \quad \Omega \in \mathfrak{C}.$$

The Faber–Krahn, Krahn–Szegö, and Kohler–Jobin inequalities are of this form. See, for example, the seminal text [11]. In a recent paper [3], a more general set of inequalities was investigated. These are of the following form: let $q \in \mathbb{R}$, and consider the shape functional

$$G(\Omega) = H(\Omega)F(\Omega)^q.$$

Then, unless $q = -\beta_2/\beta_1$, this product is not scaling invariant. However, denoting by $|\Omega|$ the Lebesgue measure of Ω, the quantity

$$\frac{H(\Omega)F(\Omega)^q}{|\Omega|^{(\beta_2+q\beta_1)/d}}$$

is scaling invariant. The case where H is the principal Dirichlet eigenvalue, and F is the torsional rigidity was analysed in [3]. In the present paper, we investigate, in the spirit of [11],

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the case where H is the Newtonian capacity, and F is the torsional rigidity. Since the Newtonian capacity is most easily defined for compact subsets of $\mathbb{R}^d$, $d \geqslant 3$, we restrict ourselves to open sets $\Omega \subset \mathbb{R}^d$, $d \geqslant 3$ which are precompact. In that case, the Newtonian capacity scales as a power $\beta_2 = d-2$ of the homothety.

Throughout this paper we let Ω be a non-empty, open, bounded set in Euclidean space $\mathbb{R}^d$, $d \geqslant 3$. For a set $A \subset \mathbb{R}^d$, we denote by $\overline{A}$ its closure, $\mathrm{diam}(A) = \sup\{|x-y| \ : \ x \in A, \ y \in A\}$ its diameter, and $r(A) = \sup\{r \geqslant 0 \ : \ (\exists x \in A), \ (B_r(x) \subset A)\}$ its inradius, where $B_r(x) = \{y \in \mathbb{R}^d \ : \ |x-y| < r\}$ is the ball of radius r centred at x. Before we state the main results we recall some basic facts about the torsion function, torsional rigidity, and Newtonian capacity.

The torsion function for an open set Ω with finite measure is the solution of

$$-\Delta u = 1, \quad u \in H_0^1(\Omega),$$

and is denoted by u_Ω. It is convenient to extend u_Ω to all of $\mathbb{R}^d$ by defining $u_\Omega = 0$ on $\mathbb{R}^d \setminus \Omega$. It is well known that u_Ω is non-negative, bounded ([2, 4 6, 12]), and monotone increasing with respect to Ω, that is

$$\Omega_1 \subset \Omega_2 \Rightarrow u_{\Omega_1} \leqslant u_{\Omega_2}.$$

The torsional rigidity of Ω, or torsion for short, is denoted by

$$T(\Omega) = \|u_\Omega\|_1,$$

where $\|\cdot\|_p$, $1 \leqslant p \leqslant \infty$ denotes the usual L^p norm. It follows that

$$\Omega_1 \subset \Omega_2 \Rightarrow T(\Omega_1) \leqslant T(\Omega_2), \tag{1}$$

and that the torsion satisfies the scaling property

$$T(t\Omega) = t^{d+2} T(\Omega), \quad t > 0. \tag{2}$$

Moreover T is additive on unions of disjoint families of open sets:

$$T(\cup_{i\in I}\Omega_i) = \sum_{i\in I} T(\Omega_i).$$

It is straightforward to verify that if $E(a)$, with $a = (a_1, a_2, \dots, a_d) \in \mathbb{R}_+^d$, is the ellipsoid

$$E(a) = \left\{ x \in \mathbb{R}^d \ : \ \sum_{i=1}^d \frac{x_i^2}{a_i^2} < 1 \right\},$$

then

$$u_{E(a)}(x) = \frac{1}{2}\left(\sum_{i=1}^d \frac{1}{a_i^2}\right)^{-1}\left(1 - \sum_{i=1}^d \frac{x_i^2}{a_i^2}\right),$$

and

$$T(E(a)) = \frac{\omega_d}{d+2}\left(\prod_{i=1}^d a_i\right)\left(\sum_{i=1}^d \frac{1}{a_i^2}\right)^{-1}, \tag{3}$$

where

$$\omega_d = \frac{\pi^{d/2}}{\Gamma((d+2)/2)}$$

is the Lebesgue measure of a ball B_1 with radius 1 in $\mathbb{R}^d$. We put

$$\tau_d = T(B_1) = \frac{\omega_d}{d(d+2)}.$$

The de Saint-Venant inequality (see, for instance, [11, Chapter V]) asserts that

$$T(\Omega) \leqslant T(\Omega^*), \tag{4}$$

where Ω^* is any ball with $|\Omega| = |\Omega^*|$. It follows by scaling that

$$\frac{T(\Omega)}{|\Omega|^{(d+2)/d}} \leqslant \frac{\tau_d}{\omega_d^{(d+2)/d}} = \frac{1}{d(d+2)\omega_d^{2/d}}. \tag{5}$$

Below we recall some basic facts about the Newtonian capacity $\mathrm{cap}\,(K)$ of a compact set $K \subset \mathbb{R}^d$, $d \geqslant 3$. There are several equivalent definitions of $\mathrm{cap}\,(K)$ of which we choose

$$\mathrm{cap}\,(K) = \inf\left\{\int_{\mathbb{R}^d} |D\varphi|^2\,dx \ : \ \varphi_{K_\varepsilon} \geqslant 1,\ \varphi \in C_0^1(\mathbb{R}^d), \varepsilon > 0\right\},$$

where φ_{K_ε} is the restriction of φ to $K_\varepsilon = \{x \in \mathbb{R}^d : \mathrm{dist}(x, K) < \varepsilon\}$. It follows that

$$K_1 \subset K_2 \Rightarrow \mathrm{cap}\,(K_1) \leqslant \mathrm{cap}\,(K_2), \tag{6}$$

and that the capacity satisfies the scaling property

$$\mathrm{cap}\,(tK) = t^{d-2}\mathrm{cap}\,(K),\ t > 0. \tag{7}$$

Moreover if $\{K_i,\, i \in I\}$ is a countable family of compact sets such that $\cup_{i\in I} K_i$ is compact, then

$$\mathrm{cap}\,(\cup_{i\in I} K_i) \leqslant \sum_{i\in I} \mathrm{cap}\,(K_i).$$

It was reported in [8, pp. 260] that the Newtonian capacity of an ellipsoid was computed in volume 8, [5, pp. 103–104]. The formula is given in terms of an elliptic integral, and reads

$$\mathrm{cap}\left(\overline{E(a)}\right) = \frac{4\pi^{d/2}}{\Gamma(d/2)}\mathfrak{e}(a)^{-1}, \tag{8}$$

where

$$\mathfrak{e}(a) = \int_0^\infty \left(\prod_{i=1}^d \left(a_i^2 + t\right)\right)^{-1/2} dt. \tag{9}$$

We put

$$\kappa_d = \mathrm{cap}\,(\overline{B_1}) = \frac{4\pi^{d/2}}{\Gamma((d-2)/2)},$$

so that

$$\mathrm{cap}\left(\overline{E(a)}\right) = \frac{2\kappa_d}{d-2}\mathfrak{e}(a)^{-1}. \tag{10}$$

The isoperimetric inequality for Newtonian capacity (see [11]) asserts that for all compact sets $K \subset \mathbb{R}^d$, $d \geqslant 3$,

$$\mathrm{cap}\,(K) \geqslant \mathrm{cap}\,(K^*),$$

where K^* is a closed ball with $|K| = |K^*|$. It follows by scaling that

$$\frac{\mathrm{cap}\,(K)}{|K|^{(d-2)/d}} \geqslant \frac{\kappa_d}{\omega_d^{(d-2)/d}}. \tag{11}$$

The shape functional we consider in the present paper is

$$G_q(\Omega) = \frac{\mathrm{cap}\,(\overline{\Omega})T(\Omega)^q}{|\Omega|^{1+q+2(q-1)/d}}, \tag{12}$$

defined for a bounded open set $\Omega\subset\mathbb{R}^d$, $d\geqslant 3$. By (2) and (7), we obtain that G_q is scaling invariant. With the definitions above, we have

$$G_q(B_1)=\frac{\kappa_d\tau_d^q}{\omega_d^{1+q+2(q-1)/d}}.$$

Since the ball Ω^* with measure $|\Omega^*|=|\Omega|$ maximises the torsional rigidity $T(\Omega)$ (de Saint-Venant), and its closure minimises the Newtonian capacity $\operatorname{cap}(\overline{\Omega})$, competition enters in the minimisation or maximisation problems for the functional in (12).

All of our main results are for $d\geqslant 3$, and are as follows.

Theorem 1. (i) *If $q\in\mathbb{R}$, then*

$$\sup\{G_q(\Omega)\ :\ \Omega \textit{ open and bounded}\}=+\infty.$$

(ii) *If $q\leqslant 0$, then*

$$\min\{G_q(\Omega)\ :\ \Omega \textit{ open and bounded}\}=G_q(B_1),\tag{13}$$

with equality if and only if Ω is (up to sets of capacity 0) a ball in $\mathbb{R}^d$.

(iii) *If $q>0$, then*

$$\inf\{G_q(\Omega)\ :\ \Omega \textit{ open and bounded}\}=0.$$

Theorem 2. (i) *If $q<1$, then*

$$\sup\{G_q(\Omega)\ :\ \Omega \textit{ open, bounded and convex}\}=+\infty.$$

(ii) *If $q\geqslant 1$, then*

$$\sup\{G_q(\Omega)\ :\Omega \textit{ open, bounded and convex}\}\leqslant\frac{2^{(d+2)/2}d^{3q-2+d(q+1)}}{d-2}G_q(B_1).\tag{14}$$

If $q>1$, then the variational problem in the left-hand side of (14) has a maximiser, say Ω^+. For any such maximiser,

$$\frac{\operatorname{diam}(\Omega^+)}{r(\Omega^+)}\leqslant 2d\left(\frac{2^{(d+2)/2}d^{(1+q)d+3q-2}}{d-2}\right)^{d/(2(q-1))}.\tag{15}$$

(iii) *If $q=1$ and $d=3$, then the variational problem in the left-hand side of (14) has a maximiser, say Ω^+. For any such maximiser,*

$$\frac{\operatorname{diam}(\Omega^+)}{r(\Omega^+)}\leqslant 2\cdot 3^8e^{3^7}.\tag{16}$$

Theorem 3. (i) *If $q>(d-2)/(2(d-1))$, then*

$$\inf\{G_q(\Omega)\ :\ \Omega \textit{ open, bounded and convex}\}=0.\tag{17}$$

(ii) *If $0<q\leqslant(d-2)/(2(d-1))$, then*

$$\inf\{G_q(\Omega)\ :\ \Omega \textit{ open, bounded and convex}\}\geqslant\frac{1}{2d^{d+(d+2)q}}G_q(B_1).\tag{18}$$

If $0<q<(d-2)/(2(d-1))$, then the variational problem in the left-hand side of (18) has a convex minimiser, say Ω^-. For any such minimiser,

$$\frac{\operatorname{diam}(\Omega^-)}{r(\Omega^-)}\leqslant 2d\left(2d^{d+(d+2)q}\right)^{\frac{d(d-1)}{d-2-2q(d-1)}}.\tag{19}$$

We were unable to prove the existence or non-existence of a maximiser for the left-hand side of (14) for $q = 1$ and $d > 3$. In these higher dimensional cases, there is a lack of compactness. For example, if

$$a_\varepsilon = (\underbrace{1, \dots, 1}_{d-k}, \underbrace{\varepsilon, \dots, \varepsilon}_{k}),$$

and if $k \geqslant 3$, then $\lim_{\varepsilon\to 0} G_1(E(a_\varepsilon))$ exists and is strictly positive. Similarly we were unable to prove the existence of a minimiser of the left-hand side of (18) at the critical point $q = (d-2)/(2d-2)$.

The proofs of Theorems 1–3 are deferred to Sections 2–4, respectively. A key ingredient in these proofs is John's ellipsoid theorem [**9**]. This theorem asserts that for any open, bounded convex set Ω in $\mathbb{R}^d$ there exists a translation and rotation of Ω, again denoted by Ω, and an open ellipsoid $E(a)$ such that

$$E(a/d) \subset \Omega \subset E(a). \tag{20}$$

Moreover, among all ellipsoids in Ω, $E(a/d)$ has maximal measure.

Finally in Section 5 we discuss the optimisation of a functional over all open bounded planar convex sets with fixed measure, and which involves the logarithmic capacity and torsional rigidity.

2. *Proof of Theorem* 1

Proof. To prove the assertion under (i), we let Ω be the disjoint union of an open ball B' of measure $1/2$ and an open ellipsoid $E(b_\varepsilon)$, with $b_\varepsilon = (L_\varepsilon, \dots, L_\varepsilon, \varepsilon)$, of measure $1/2$, where

$$L_\varepsilon = (2\omega_d\varepsilon)^{1/(1-d)}.$$

We have, by (8) and (9),

$$\begin{aligned}
\mathrm{cap}\left(\overline{E(b_\varepsilon)}\right) &= \frac{4\pi^{d/2}}{\Gamma(d/2)}\left(\int_0^\infty dt\,(L_\varepsilon^2+t)^{(1-d)/2}(\varepsilon^2+t)^{-1/2}\right)^{-1} \\
&\geqslant \frac{4\pi^{d/2}}{\Gamma(d/2)}\left(\int_0^\infty dt\,(L_\varepsilon^2+t)^{(1-d)/2}t^{-1/2}\right)^{-1} \\
&= \frac{4\pi^{(d-1)/2}\Gamma((d-1)/2)}{\Gamma(d/2)\Gamma((d-2)/2)}L_\varepsilon^{d-2},
\end{aligned}$$

where we have used formulae [**7**, 8.380.3 and 8.384.1]. Hence

$$\begin{aligned}
G_q(\Omega) &= \mathrm{cap}\left(\overline{B'} \cup \overline{E(b_\varepsilon)}\right)T(B' \cup E(b_\varepsilon))^q \\
&\geqslant \mathrm{cap}\left(\overline{E(b_\varepsilon)}\right)T(B')^q \\
&\geqslant \frac{4\pi^{(d-1)/2}\Gamma((d-1)/2)}{\Gamma(d/2)\Gamma((d-2)/2)}T(B')^q(2\omega_d\varepsilon)^{(d-2)/(1-d)},
\end{aligned}$$

which tends to $+\infty$ as $\varepsilon \downarrow 0$.

To prove the assertion under (ii), we recall (4), and infer that $T^q(\Omega) \geqslant T^q(\Omega^*)$ for $q \leqslant 0$. This implies (13) by (5) and (11).

To prove (iii), we let $Q \subset \mathbb{R}^d$ be a cube with $|Q| = 1$. Let $N \in \mathbb{N}$ be arbitrary. The cube Q contains N^d open disjoint cubes each of measure N^{-d}. Each open cube contains an open ball with radius $1/(2N)$. Let Q_N be the union of these N^d open balls. Since $Q_N \subset Q$ we have $\mathrm{cap}\,(\overline{Q_N}) \leqslant \mathrm{cap}\,(\overline{Q})$. On the other hand, additivity and scaling properties of the torsion give

$$T(Q_N) = N^d(2N)^{-(d+2)}T(B_1) = 2^{-d-2}N^{-2}T(B_1).$$

Furthermore,

$$|Q_N| = \frac{\omega_d}{2^d}.$$

Hence

$$\begin{aligned}\inf\{G_q(\Omega)\ :\ \Omega \text{ open and bounded}\} &\leqslant \frac{\mathrm{cap}\,(\overline{Q})T^q(Q_N)}{|Q_N|^{1+q+2(q-1)/d}}\\ &= \frac{2^{d-2}}{\omega_d^{1+q+2(q-1)/d}}\mathrm{cap}\,(\overline{Q})T^q(B_1)N^{-2q}.\end{aligned}$$

This implies (17) since $q > 0$, and $N \in \mathbb{N}$ was arbitrary. □

3. *Proof of Theorem 2*

Proof. To prove (i) we consider the open ellipsoid $E(a_\varepsilon)$ with $a_\varepsilon = (1, \varepsilon, \ldots, \varepsilon)$. We have

$$|E(a_\varepsilon)| = \omega_d \varepsilon^{d-1},$$

$$T(E(a_\varepsilon)) = \frac{\omega_d}{d+2}\frac{\varepsilon^{d+1}}{d-1+\varepsilon^2},$$

$$\mathrm{cap}\left(\overline{E(a_\varepsilon)}\right) = \begin{cases} 4\pi\varepsilon\big(\log(\varepsilon^{-1})\big)^{-1}(1+o(1)), & d=3,\ \varepsilon\downarrow 0,\\ \dfrac{2\pi^{d/2}(d-3)}{\Gamma(d/2)}\varepsilon^{d-3}(1+o(1)), & d>3,\ \varepsilon\downarrow 0,\end{cases}$$

where we have used the formulae on [8, p. 260]. Hence

$$G_q(E(a_\varepsilon)) = \begin{cases} C_3\varepsilon^{2(q-1)/3}\big(\log\varepsilon^{-1}\big)^{-1}(1+o(1)), & d=3,\ \varepsilon\downarrow 0,\\ C_d\varepsilon^{2(q-1)/d}(1+o(1)), & d>3,\ \varepsilon\downarrow 0,\end{cases}$$

where C_d is a positive constant depending only on d. Since $q < 1$, we obtain the desired result by letting $\varepsilon \downarrow 0$.

To prove (ii), we first observe that the formulae for $|E(a)|, \mathrm{cap}\,(\overline{E(a)})$, and $T(E(a))$ are symmetric in $a_1, \ldots, a_d$. Without loss of generality we may therefore assume here, and throughout this paper, that $a_1 \geqslant a_2 \geqslant \ldots \geqslant a_d$. By inclusion, and (20) we have

$$d^{-d}\omega_d\prod_{i=1}^d a_i = |E(a/d)| \leqslant |\Omega| \leqslant |E(a)| = \omega_d\prod_{i=1}^d a_i, \tag{21}$$

and

$$T(\Omega) \geqslant T(E(a/d)) = \frac{\omega_d}{d^{d+2}(d+2)}\left(\prod_{i=1}^d a_i\right)\left(\sum_{i=1}^d \frac{1}{a_i^2}\right)^{-1} \geqslant \frac{\tau_d}{d^{d+2}}\left(\prod_{i=1}^d a_i\right)a_d^2. \tag{22}$$

We have by (8),

$$\mathfrak{e}(a) \geqslant \int_0^{a_d^2} dt\left(\prod_{i=1}^d (a_i^2+t)\right)^{-1/2}$$

$$\geqslant a_d^2\left(\prod_{i=1}^{d}\left(a_i^2+a_d^2\right)\right)^{-1/2}$$

$$\geqslant 2^{-d/2}a_d^2\left(\prod_{i=1}^{d}a_i^2\right)^{-1/2}$$

$$= 2^{-d/2}\left(\prod_{i=1}^{d}a_i\right)^{-1}a_d^2. \tag{23}$$

By (9), (10) and (23), taking into account that $\Gamma(z+1)=z\Gamma(z)$, $z>0$,

$$\operatorname{cap}(\overline{\Omega})\leqslant \operatorname{cap}(\overline{E(a)})\leqslant \frac{2^{(d+2)/2}\kappa_d}{d-2}\left(\prod_{i=1}^{d}a_i\right)a_d^{-2}. \tag{24}$$

By (1) and (3),

$$T(\Omega)\leqslant T(E(a))\leqslant d\tau_d\left(\prod_{i=1}^{d}a_i\right)a_d^2. \tag{25}$$

By (21), (24), (25), $a_1\geqslant a_2\geqslant\ldots\geqslant a_d$, and $q>1$, we obtain,

$$\begin{aligned}
G_q(\Omega)&\leqslant \frac{\operatorname{cap}(\overline{E(a)})T(E(a))^q}{|E(a/d)|^{1+q+2(q-1)/d}}\\
&\leqslant \frac{2^{(d+2)/2}\kappa_d}{d-2}(d\tau_d)^q\left(d^{-d}\omega_d\right)^{-(1+q+2(q-1)/d)}\left(\prod_{i=1}^{d}a_i\right)^{2(1-q)/d}a_d^{2q-2}\\
&=\frac{2^{(d+2)/2}d^{3q-2+d(q+1)}}{d-2}G_q(B_1)\left(\prod_{i=1}^{d}a_i\right)^{2(1-q)/d}a_d^{2q-2}\\
&\leqslant \frac{2^{(d+2)/2}d^{3q-2+d(q+1)}}{d-2}G_q(B_1).
\end{aligned} \tag{26}$$

This proves (14).

To prove the existence of a maximiser, we observe that if the left-hand side of (14) equals $G_q(B_1)$, then B_1 is a maximiser which satisfies (15). If the left-hand side of (14) is strictly greater than $G_q(B_1)$, we let Ω be bounded, open, and convex, and such that

$$G_q(\Omega)>G_q(B_1). \tag{27}$$

By the third inequality in (26), (27), $q>1$, and $a_1\geqslant a_2\geqslant\ldots\geqslant a_d$, we find that

$$a_1\leqslant \beta_d^{d/(2(q-1))}a_d, \tag{28}$$

where β_d is the coefficient of $G_q(B_1)$ in the right-hand side of (26). Since

$$\operatorname{diam}(\Omega)\leqslant \operatorname{diam}(E(a))\leqslant 2a_1, \tag{29}$$

and

$$r(\Omega)\geqslant r(E(a/d))=\frac{a_d}{d}, \tag{30}$$

we obtain by (28)–(30),

$$\frac{\operatorname{diam}(\Omega)}{r(\Omega)} \leqslant 2d\left(\frac{2^{(d+2)/2}d^{3q-2+d(q+1)}}{d-2}\right)^{d/(2(q-1))}. \tag{31}$$

Let (Ω_n) be a maximising sequence for the left-hand side of (14). Since this supremum is scaling invariant, we fix $r(\Omega_n)=1$. By (31), $\operatorname{diam}(\Omega_n) \leqslant L$ for some $L<\infty$, and for all n. By taking translations of (Ω_n), these translates are contained in a closed ball B_L of radius L. Since the Hausdorff metric is compact on the space of convex, compact sets in B_L, there exists a subsequence of $(\overline{\Omega_n})$, again denoted by $(\overline{\Omega_n})$ which converges in the Hausdorff (and in the complementary Hausdorff) metric to an element say $\tilde{\Omega}^+$. Set $\Omega^+ = \operatorname{int}(\tilde{\Omega}^+)$. Note that Ω^+ is an open, bounded, convex set which is non-empty since Ω^+ has inradius 1. Furthermore measure, torsion, capacity, and diameter are all continuous with respect to this metric. Hence

$$G_q(\Omega^+) = \lim_{n\to\infty} G_q(\Omega_n),$$

and Ω^+ is a maximiser which satisfies (15).

To prove (iii), we let $q=1$ and $d=3$. Let Ω be an element of a maximising sequence. We may assume that

$$G_1(B_1) \leqslant G_1(\Omega) \leqslant \frac{\operatorname{cap}\left(\overline{E(a)}\right)T(E(a))}{\left|E(a/3)\right|^2}. \tag{32}$$

We obtain an upper bound on $\operatorname{cap}(\overline{E(a)})$ by obtaining a lower bound on $\mathfrak{e}(a)$. By (9), we have

$$\begin{aligned}
\mathfrak{e}(a_1,a_2,a_3) &\geqslant \int_0^\infty \left(a_1^2+t\right)^{-1/2}\left(a_2^2+t\right)^{-1}dt \\
&= \frac{2}{\left(a_1^2-a_2^2\right)^{1/2}}\log\left(\frac{a_1}{a_2}+\left(\frac{a_1^2}{a_2^2}-1\right)^{1/2}\right) \\
&\geqslant \frac{2}{a_1}\log\left(a_1a_2^{-1}\right).
\end{aligned} \tag{33}$$

By (8) for $d=3$, and (33),

$$\operatorname{cap}(\overline{E(a)}) \leqslant \kappa_3 a_1\left(\log\left(a_1a_2^{-1}\right)\right)^{-1}.$$

Since $\Omega \subset B_{a_1}$, we also have

$$\operatorname{cap}(\overline{\Omega}) \leqslant \operatorname{cap}(\overline{B_{a_1}}) = \kappa_3 a_1.$$

Hence

$$\operatorname{cap}(\overline{\Omega}) \leqslant \kappa_3 a_1 \min\left\{1, \left(\log\left(a_1a_2^{-1}\right)\right)^{-1}\right\}.$$

In addition,

$$T(E(a)) \leqslant 3\tau_3 a_1a_2a_3^3\,, \quad |E(a/3)| = 3^{-3}\omega_3 a_1a_2a_3\,.$$

Summarising, from (32) we obtain

$$G_1(\Omega) \leqslant 3^7 G_1(B_1)\cdot\frac{a_3}{a_2}\cdot\min\{1,\left(\log\left(a_1a_2^{-1}\right)\right)^{-1}\}. \tag{34}$$

If the supremum in the left-hand side of (14) equals $G_1(B_1)$, then B_1 is a maximiser which satisfies (16). If not then we may assume that $G_1(\Omega) > G_1(B_1)$. This, together with (34), yields $a_3 \geqslant 3^{-7}a_2$, $a_1 \leqslant a_2 e^{3^7}$. These inequalities imply that $a_1/a_3 \leqslant 3^7 e^{3^7}$. Hence (29) and (30) yield

$$\frac{\operatorname{diam}(\Omega)}{r(\Omega)} \leqslant 2 \cdot 3^8 e^{3^7}.$$

The remaining part of the proof follows similar lines as those in the proof of (ii). □

4. *Proof of Theorem 3*

Proof. To prove (i), we consider as Ω the ellipsoid $E(a)$ with

$$a = (a_1, \ldots, a_1, a_d), \quad a_1 \geqslant a_d, \quad a_1^{d-1} a_d = 1,$$

where $a_d \in (0,1)$ is arbitrary. Since $E(a) \subset B_{a_1}$, we have by (6), (7) and (10),

$$\operatorname{cap}(\overline{E(a)}) \leqslant \kappa_d a_1^{d-2}. \tag{35}$$

By (3) and (35),

$$G_q(E(a)) \leqslant \frac{\kappa_d a_1^{d-2}}{\omega_d^{1+q+2(q-1)/d}} \left(\frac{\omega_d}{d+2} \frac{a_1^2 a_d^2}{(d-1)a_d^2 + a_1^2} \right)^q \leqslant \frac{\kappa_d a_1^{d-2} a_d^{2q}}{\omega_d^{1+2(q-1)/d} (d+2)^q}.$$

Since $a_1 = a_d^{-1/(d-1)}$, we have

$$G_q(E(a)) \leqslant \frac{\kappa_d}{\omega_d^{1+2(q-1)/d}(d+2)^q} a_d^{-(d-2)/(d-1)+2q},$$

and since $a_d \in (0,1)$ was arbitrary, we obtain (17).

To prove (ii), we let $a_1 \geqslant a_2 \geqslant \ldots \geqslant a_d$. We have

$$\operatorname{cap}(\overline{\Omega}) \geqslant \operatorname{cap}(\overline{E(a/d)}) = d^{2-d} \operatorname{cap}(\overline{E(a)}) = \frac{2\kappa_d}{d^{d-2}(d-2)} (\mathfrak{e}(a))^{-1}. \tag{36}$$

In order to obtain an upper bound on $\mathfrak{e}(a)$, we have by using the inequality $(x^2+t)^{1/2} \geqslant 2^{-1/2}(x + t^{1/2})$, and the change of variables $t = \theta^2$,

$$\begin{aligned}
\mathfrak{e}(a) &\leqslant \left(\prod_{i \leqslant d-3} a_i^{-1} \right) \int_0^\infty dt \left((a_{d-2}^2 + t)(a_{d-1}^2 + t)(a_d^2 + t) \right)^{-1/2} \\
&\leqslant \left(\prod_{i \leqslant d-3} a_i^{-1} \right) \int_0^\infty dt \left((a_{d-2}^2 + t)(a_{d-1}^2 + t)t \right)^{-1/2} \\
&\leqslant 2 \left(\prod_{i \leqslant d-3} a_i^{-1} \right) \int_0^\infty dt \left((a_{d-2} + t^{1/2})(a_{d-1} + t^{1/2}) t^{1/2} \right)^{-1} \\
&= 4 \left(\prod_{i \leqslant d-3} a_i^{-1} \right) \int_0^\infty d\theta \left((a_{d-2} + \theta)(a_{d-1} + \theta) \right)^{-1} \\
&= 4 \left(\prod_{i \leqslant d-2} a_i^{-1} \right) \left(1 - \frac{a_{d-1}}{a_{d-2}} \right)^{-1} \log(a_{d-2}/a_{d-1}),
\end{aligned} \tag{37}$$

where the product over the empty set in the right-hand side of (37) is defined to be equal to 1, and where the case $a_{d-2} = a_{d-1}$ follows by taking the appropriate limit in the right-hand side of (37). It is elementary to verify that

$$(1-x)^{-1}\log(x^{-1}) \leqslant \log(e/x), \quad 0 < x < 1,$$

and

$$\lim_{x\uparrow 1}(1-x)^{-1}\log(x^{-1}) = 1.$$

This gives by (37),

$$\mathfrak{e}(a) \leqslant 4\left(\prod_{i\leqslant d-2} a_i^{-1}\right)\log(ea_{d-2}/a_{d-1}). \tag{38}$$

Hence by (36) and (38),

$$\mathrm{cap}\,(\overline{\Omega}) \geqslant \frac{\kappa_d}{2d^{d-2}(d-2)}\left(\prod_{i\leqslant d-2} a_i\right)\left(\log(ea_{d-2}/a_{d-1})\right)^{-1}. \tag{39}$$

By (21), (22), and (39),

$$G_q(\Omega) \geqslant \frac{1}{2d^{d-2+(d+2)q}(d-2)}G_q(B_1)\frac{a_d^{2q-1}}{a_{d-1}}\left(\prod_{i=1}^{d} a_i\right)^{2(1-q)/d}\left(\log(ea_{d-2}/a_{d-1})\right)^{-1}. \tag{40}$$

The a-dependence in the right-hand side of (40) is scaling invariant. It is convenient to choose $\prod_{i=1}^d a_i = 1$. We then have

$$\frac{a_{d-2}}{a_{d-1}} = \left(\prod_{i\leqslant d-3} a_i^{-1}\right)a_{d-1}^{-2}a_d^{-1} \leqslant a_{d-1}^{-(d-1)}a_d^{-1}.$$

This gives with (40),

$$G_q(\Omega) \geqslant \frac{1}{2d^{d-2+(d+2)q}(d-2)}G_q(B_1)\frac{a_d^{2q-1}}{a_{d-1}}\left(\log\left(e/(a_{d-1}^{d-1}a_d)\right)\right)^{-1}. \tag{41}$$

Since

$$x^{1/(d-1)}\log(e/x) \leqslant d-1, \quad 0 < x < 1,$$

we have, with $x = a_{d-1}^{d-1}a_d$,

$$\left(\log\left(e/(a_{d-1}^{d-1}a_d)\right)\right)^{-1} \geqslant a_{d-1}a_d^{1/(d-1)}(d-1)^{-1}.$$

This, together with (41), gives

$$\begin{aligned} G_q(\Omega) &\geqslant \frac{1}{2d^{d-2+(d+2)q}(d-2)(d-1)}G_q(B_1)a_d^{2q-\frac{d-2}{d-1}} \\ &\geqslant \frac{1}{2d^{d+(d+2)q}}G_q(B_1)a_d^{2q-\frac{d-2}{d-1}}. \end{aligned} \tag{42}$$

This proves (18) since $a_d \in (0,1]$, and $q \leqslant (d-2)/(2(d-1))$.

To prove the existence of a minimiser, we observe that if the left-hand side of (18) equals $G_q(B_1)$, then B_1 is a minimiser which satisfies (19). If the left-hand side of (18) is strictly less than $G_q(B_1)$, we let Ω be bounded and convex such that

$$G_q(\Omega) < G_q(B_1). \tag{43}$$

By (42) and (43), we infer

$$a_d \geqslant \left(\frac{1}{2d^{d+(d+2)q}}\right)^{(d-1)/(d-2-2q(d-1))}. \tag{44}$$

Since $\prod_{i=1}^{d} a_i = 1$, and $a_1 \geqslant a_2 \geqslant \ldots \geqslant a_d$, we have $a_1 \leqslant a_d^{1-d}$. By (29), (30) and (44), we obtain

$$\frac{\mathrm{diam}(\Omega)}{r(\Omega)} \leqslant 2da_d^{-d} \leqslant 2d\Big(2d^{d+(d+2)q}\Big)^{\frac{d(d-1)}{d-2-2q(d-1)}}. \tag{45}$$

The proof of the existence of a minimiser is similar to the proof of the existence of a maximiser in Theorem 1(ii), and has been omitted. If Ω^- is a minimiser, then, by continuity of diameter and inradius, Ω^- satisfies (45). This proves (19). □

5. *The logarithmic capacity*

We briefly recall some basic properties of the logarithmic capacity of a compact set K in $\mathbb{R}^2$. Let μ be a probability measure supported on K, and let

$$I(\mu) = \iint_{K\times K} \log\left(\frac{1}{|x-y|}\right)\mu(dx)\mu(dy).$$

Furthermore let

$$V(K) = \inf\{I(\mu) : \mu \text{ a probability measure on } K\}.$$

The logarithmic capacity of K is denoted by $\mathrm{cap}\,(K)$, and is the non-negative real number

$$\mathrm{cap}\,(K) = e^{-V(K)}.$$

It shares some of the properties of the Newtonian capacity. In particular, if K_1 and K_2 are compact sets in $\mathbb{R}^2$ with $K_1 \subset K_2$, then $\mathrm{cap}\,(K_1) \leqslant \mathrm{cap}\,(K_2)$. Moreover, $\mathrm{cap}\,(K)$ is invariant under translations and rotations of K, and

$$\mathrm{cap}\,(K) \geqslant \mathrm{cap}\,(K^*), \tag{46}$$

where K^* is the disc with $|K| = |K^*|$. See [1] for some refinements. Finally for a homothety,

$$\mathrm{cap}\,(tK) = t\,\mathrm{cap}\,(K), \quad t > 0. \tag{47}$$

The classic treatise [10] gives various planar domains for which the logarithmic capacity can be computed analytically. In particular, for the ellipse with semi axes a_1 and a_2,

$$\mathrm{cap}\,(\overline{E(a_1,a_2)}) = \frac{1}{2}(a_1 + a_2). \tag{48}$$

For an open, bounded, convex planar set Ω, we define the functional

$$H_q(\Omega) = \frac{\mathrm{cap}\,(\overline{\Omega})T^q(\Omega)}{|\Omega|^{(1+4q)/2}}.$$

In particular, we have

$$H_q(B_1) = \frac{\tau_2^q}{\omega_2^{(1+4q)/2}} = \frac{1}{8^q\pi^{q+1/2}}.$$

We immediately see that by (2), and (47) that $H_q(t\Omega) = H_q(\Omega)$, $t > 0$. Our main result is the following.

THEOREM 4. (i) *If $q \geqslant 1/2$, then*

$$\sup\{H_q(\Omega) \ : \ \Omega \text{ open, bounded, planar, and convex}\} \leqslant 2^{1+5q} H_q(B_1). \tag{49}$$

(ii) *If $q > 1/2$, then the left-hand side of (49) has an open, bounded, planar, and convex maximiser. For any such maximiser, say Ω^+,*

$$\frac{\mathrm{diam}(\Omega^+)}{r(\Omega^+)} \leqslant \frac{2^{14q}}{2q-1}. \tag{50}$$

(iii) *If $q < 1/2$, then*

$$\sup\{H_q(\Omega) \ : \ \Omega \text{ open, bounded, planar, and convex}\} = +\infty. \tag{51}$$

(iv) *If $q \leqslant 1/2$, then*

$$\inf\{H_q(\Omega) \ : \ \Omega \text{ open, bounded, planar, and convex}\} \geqslant 2^{-2(1+2q)} H_q(B_1). \tag{52}$$

(v) *If $q < 1/2$, then the left-hand side of (52) has an open, bounded, planar, and convex minimiser. For any such minimiser, say Ω^-,*

$$\frac{\mathrm{diam}(\Omega^-)}{r(\Omega^-)} \leqslant 2^{2(3+2q)/(1-2q)}. \tag{53}$$

(vi) *If $q > 1/2$, then*

$$\inf\{H_q(\Omega) \ : \ \Omega \text{ open, bounded, planar, and convex}\} = 0.$$

Proof. (i) If $E(a)$ is the John's ellipsoid for Ω, then $E(a/2) \subset \Omega \subset E(a)$ with $a_1 \geqslant a_2$. Furthermore,

$$\mathrm{cap}\,(\overline{E(a)}) \leqslant a_1, \quad T(E(a)) \leqslant 2\tau_2 a_1 a_2^3, \quad |\Omega| \geqslant |E(a/2)| = \omega_2 a_1 a_2/4,$$

so that

$$H_q(\Omega) \leqslant 2^{1+5q} \frac{\tau_2^q}{\omega_2^{(1+4q)/2}} \left(\frac{a_2}{a_1}\right)^{q-1/2}. \tag{54}$$

This implies (49) since $q \geqslant 1/2$.

(ii) To prove (50), we have that either the supremum in the left-hand side of (49) is attained for a ball, in which case the maximiser exists and satisfies (50), or we may assume that $H_q(\Omega) > H_q(B_1)$. This implies, by (29) and (30),

$$\frac{\mathrm{diam}(\Omega)}{r(\Omega)} \leqslant \frac{2^{14q}}{2q-1}.$$

The remaining part of the proof is similar to the corresponding parts in the proof of Theorem 2.

(iii) By (21), (22), and (48),

$$H_q(\Omega) \geqslant 2^{-2(1+2q)} \frac{\tau_2^q}{\omega_2^{(1+4q)/2}} \left(\frac{a_2}{a_1}\right)^{q-1/2}. \tag{55}$$

This implies (51) by letting $a_2/a_1 \to 0$ in (55).

(iv) This follows from (55) and $a_1 \geqslant a_2$.

(v) Either the infimum in the left-hand side of (52) is attained for a ball, in which case the minimiser exists and satisfies (53), or we may assume that $H_q(\Omega) < H_q(B_1)$. By (55), (29), and (30)

$$\frac{\mathrm{diam}(\Omega)}{r(\Omega)} \leqslant 2^{\frac{2(3+2q)}{1-2q}}.$$

The remaining part of the proof is similar to the corresponding parts in the proof of Theorem 2.

(vi) This follows by letting $a_2/a_1 \to 0$ in (54). ∎

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